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Determination of Dominant Error Sources in an Inertial Navigation System by Iterative Weighted Least Squares

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A critical problem in inertial navigation system evaluation is the determination of the dominant system error sources by an analysis of flight test data For both cruise and ballistic systems, this problem is reduced to estimating the coefficients in a linear model from data corrupted by nonstationary noise A procedure is presented which estimates the coefficients when the noise is nonstationary and/or correlated The procedure estimates the coefficients by least squares; then it iteratively obtains estimates of the necessary noise variances and covariances, and uses this information to re-estimate the coefficients by weighted least squares Two IBM 7090 or 7094 FORTRAN computer programs have been written to implement the procedure for nonstationary, uncorrelated noise An illustrative example is included

Introduction

THE use of inertial guidance equipment may conveniently be divided into two major areas, namely, cruise applications and ballistic applications Cruise systems are characterized by operating times that are significant compared to the Schuler 84-min period and, generally, are subjected to relatively low accelerations Typical applications in navigation are for aircraft, ships, submarines, and cruise (air-breathing) missiles Ballistic systems are characterized by shorter flight times compared to the Schuler period (i e , 5 min or less) and relatively high accelerations The inertial system error sources, which act as the dominant factors in determining system performance, differ in the two applications In cruise systems, gyro errors such as random drift and mass unbalance dominate In ballistic systems, uncertainty of accelerometer scale factor and accelerometer nonlinearities tend to be the dominant errors A critical problem in inertial navigation system evaluation is, then, the determination of the dominant system error sources

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Ultimately, there is no substitute for flight testing of inertial systems Laboratory tests and such techniques as centrifuge testing and sled testing can supply much useful information; however, effects of sustained linear accelerations, aircraft maneuvers, complex vibration, and the acoustic environment acting upon the entire system cannot be adequately duplicated on the ground

The following paragraphs briefly discuss the error models for cruise and ballistic systems and present a procedure to determine the dominant system error sources by an analysis of flight test data

Cruise System Error Model

It has been shown[1,2] that the propagation of the position error of a cruise autonavigator is given by

$$(d^2\overline{\Delta R}/dt^2) + 2\bar{\Omega} \times (d\overline{\Delta R}/dt) + \omega^2\overline{\Delta R} = \bar{D} \qquad (1)$$

where $\overline{\Delta R}$ is the vehicle position error vector, ω is the Schuler frequency, $[\omega = (g/a)^{1/2}]$ $\bar{\Omega}$ is the earth angular velocity vector, $\bar{D}$ is the error driving function, and a is the radius of the earth $\bar{D}$ may be generated by such error sources as initial platform misalignment, gyro drift, initial position error, and accelerometer bias shift and scale factor errors For shorter operating times compared to the 24-hr period of the earth's rotation, the Coriolis term in Eq (1), $2\bar{\Omega} \times (d\overline{\Delta R}/dt)$, is usually neglected

In general, as a result of flight test, position error of the autonavigator as a function of time, $\overline{\Delta R}(t)$, has been determined by comparing onboard system readouts with the test range instrumentation or other external position references It is now desired to determine the dominant error sources from these data A possible approach is to attempt a direct solution of Eq (1) for $\bar{D}$ and then to deduce the error sources from the driving function Ostensibly, this approach appears to offer the following significant advantages: 1) correlation between error driving function and vehicle acceleration may be readily seen and acceleration-dependent error sources determined; and 2) error sources that may start propagation at some arbitrary time during the flight (e g , bias shifts) can be quickly found Unfortunately, the method just mentioned suffers from several drawbacks The major drawback results because noise in the position error data $\overline{\Delta R}$ (t), even after heavy smoothing, is amplified when the first and second derivatives are taken, masking the information in the driving function The method is most applicable where very high-quality external data are available and where operating times are long, at least compared to the Schuler period

To avoid the problems inherent in taking the first and second derivatives of noisy data, the observed position error propagation may be expressed as a linear combination of the propagations of the individual error sources and the resulting set of equations solved for the error source magnitudes Thus, the measured autonavigator position error in, say, the northerly direction, ΔN, at a given time t_i is

$$\Delta N_i = \beta_1 X_{1i} + \beta_2 X_{2i} + \quad + \beta_p X_{pi} \tag{2}$$

where β_1, β_2, , β_p are the error parameters to be determined and X_{1i}, X_{2i}, , X_{pi} are the coordinate functions (evaluated at time t_i) that express the time propagation of the error sources In writing Eq (2), assumptions generally made are that 1) the autonavigator error model is known a priori; 2) the error parameters β_j remain constant over the final phase of alignment and during the navigation mode of operation; and 3) alignment has progressed sufficiently before switching to the navigation mode that all errors have damped to their steady-state propagation The free-inertial error propagation X_j for a given error source may be obtained by solution of Eq (1); the correlated error arising out of alignment is derived from a consideration of the system alignment error transfer functions The explicit functional forms for X_j are developed in Ref 2

Equation (2) must often be solved in the face of noise that corrupts the quality of the position error information For cruise systems, the craft may be successively tracked by various radars of differing accuracy; furthermore, the accuracy of the radar is range-dependent Thus, the resulting noise is nonstationary in character A similar situation exists in the cases where external reference data are obtained from LORAN information or the use of an airborne camera Because many of the mathematical techniques for extracting information from noisy data are based upon the assumption of stationarity (e g , classical least squares), somewhat more sophisticated methods are required As will be indicated in succeeding paragraphs, a similar problem exists for ballistic system range instrumentation

A further complicating factor is the high correlation of several of the error sources in Eq (2) Thus, the attempted solution of Eq (2), depending upon the error sources admitted to the model, often results in matrices that are near-singular, with the consequence that roundoff error dominates the solution In this regard, matters can often be greatly improved by a judicious choice of coordinate system As an excellent example of this, consider the case of error propagation of initial azimuth misalignment and constant gyro drift

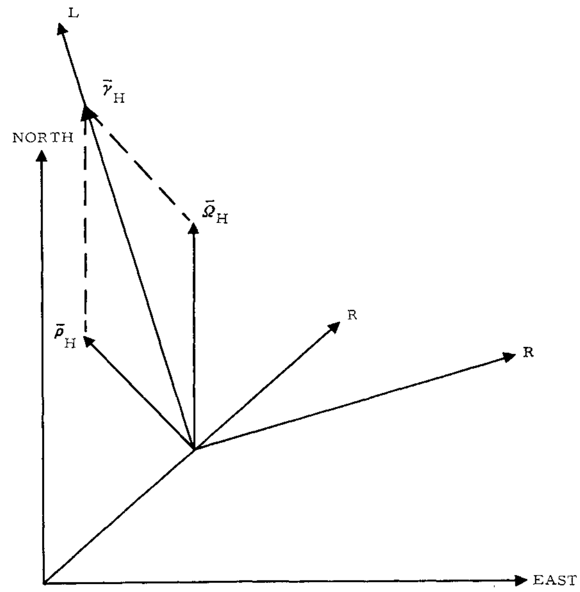

Fig 1 Pseudo-range and lateral coordinates

for inflight aligned cruise systems For flight times less than several hours, when viewed in platform coordinates, these two error sources are so highly correlated as to be virtually inseparable It can be shown, however, that the position error $\overline{\Delta P}$ (t) that propagates as a result of an initial azimuth misalignment is given by

$$\overline{\Delta P}\ (t) = a\psi f(t)\ \bar{\gamma}_H \tag{3}$$

where ψ is the initial azimuth misalignment, a is the equatorial radius of the earth, $f(t)$ is a function dependent only upon time, and $\bar{\gamma}_H$ is the horizontal component of the total platform torquing rate vector The entire position error due to an azimuth misalignment is thus in a direction defined by $\bar{\gamma}_H$ (L', Fig 1) This is shown graphically in Fig 1, where R is the direction of the horizontal velocity vector, $\bar{\rho}_H$ is the angular rate with respect to the earth, and $\bar{\Omega}_H$ is the horizontal component of the earth's angular rate at the particular location Regression analysis of the position error propagation in the L' and R' channels will result in the quantity $\gamma_H\psi + \epsilon_R$ being obtained from the pseudolateral channel (where ϵ_R is the system gyro drift measured about the R' axis and γ_H the horizontal component of the total spatial rate vector) and ϵ_L alone being obtained from the pseudo range channel ‡ Over an ensemble of flight tests, the mean-squared values of ϵ_R and ϵ_L are equal, that is,

$$\langle \epsilon_R{}^2 \rangle = \langle \epsilon_L{}^2 \rangle \tag{4}$$

where the angular brackets represent ensemble averages Furthermore, since ψ and ϵ_R are independent and have zero average values,

$$\langle (\gamma_H \psi + \epsilon_R)^2 \rangle = \gamma_H{}^2 \langle \psi^2 \rangle + \langle \epsilon_R{}^2 \rangle \tag{5}$$

Thus, given an ensemble of flight tests, by a fortunate choice of coordinate system, the ensemble characteristics of two very highly correlated error sources can be separated

Ballistic System Error Model

In an inertial guidance system for ballistic vehicles, the most significant error sources relate to the accelerometers, that is, the errors in mechanizing the accelerometer linear scale factor and the higher-order terms in the scale factor model which are not mechanized Typically, the ballistic

‡ The pseudocoordinate system described was originally suggested by R F Nease

system acceleration error model is

$$\Delta A = \beta_1 A_{\text{in}} + \beta_2 A_{\text{in}}^2 + \beta_3 A_{\text{in}}^3 + \quad + \beta_4 A_{\text{in}} A \quad + \beta_5 A_{\text{in}}^2 A \quad + \tag{6}$$

where A_{in} and A are the accelerometer input-axis and cross-axis accelerations, respectively; the β's are the unknown scale factors and are to be determined from flight test Integrating Eq (6) to obtain the velocity error propagation,

$$\Delta V(t) = \beta_1 \int_{t_0}^{t} A_{\text{in}} d\tau + \beta_2 \int_{t_0}^{t} A_{\text{in}}^2 d\tau + \quad + \beta_4 \int_{t_0}^{t} A_{\text{in}} A_c \, d\tau + \tag{7}$$

The integrals of acceleration are obtained from a knowledge of the boost profile of the missile At a given time t_i, Eq (7) may be written

$$\Delta V_i = \beta_1 X_{1i} + \beta_2 X_{2i} + \quad + \beta_p X_{pi} \tag{8}$$

where the integrals have been replaced by the generic X_{ji}, indicating the coordinate functions [Observe the similarity of Eq (8) to Eq (2)] In Eq (8), the ΔV_i are measured, the X_{ji} are calculated, and a solution in some "best" sense is sought for the β_j

For ballistic missile error propagation, it can be demonstrated that the contribution made to impact error by position error at cutoff is small compared to the effects of velocity error Thus, error analysis for ballistic systems is generally conducted in the velocity domain, the majority of system error propagations rather severely losing their characteristic shape when viewed in the position domain Comparison on the acceleration level, badly handicapped by the need to take a second derivative of the external position information, results in data of little value for analysis even after heavy filtering is performed

The missile position as a function of time is the basic external information obtained Because the most precise ballistic missile tracking systems employ triangulation techniques, as the missile proceeds downrange the quality of the data degrades because of the increasingly unfavorable geometry Furthermore, in the latter stages of the trajectory, communication signals to the missile pass through the exhaust plume, resulting in poor transmission to the missile telemetry and instrumentation with attendant signal dropouts and increased noise The net effect for the ballistic system is similar to that for the cruise system: data corrupted by nonstationary noise, with the added disadvantage that the position data obtained by the range instrumentation must be differentiated to provide velocity data and thereby permit effective analysis The problem of high correlation within the error model[3] compounds the problem as in the cruise case

As with cruise systems, a judicious choice of coordinate system can give much insight into possible error sources For example, a transformation of the velocity error curve into coordinates normal and tangential to the thrust vector facilitates the separation of accelerometer and gyro drift errors If error in the range instrumentation is suspected, velocity comparison in the external data coordinate system can be performed to isolate a malfunctioning external data channel

To summarize, it was found that similar problems arise for both cruise and ballistic vehicle flight tests A model of the form

$$Y = \sum_{j=1}^{p} \beta_j X_j \tag{9}$$

must be estimated, where the measured error propagation Y is corrupted by nonstationary noise, and the coordinate functions X_j are often highly correlated The following paragraphs will develop a methodology for handling this problem

Consideration of the Noise Covariance Matrix§

The noise covariance matrix is the matrix of noise variances and covariances If this matrix were known, then the optimum (i e , minimum variance unbiased linear) estimators of the unknown coefficients in the error model would be the weighted least squares estimators determined by the weight matrix that is its inverse [5] Hence, it is quite reasonable to consider the estimation of the necessary noise variances and covariances as a step toward the solution of the problem

Four types of noise¶ with mean zero to be discussed are 1) stationary, uncorrelated noise; 2) nonstationary, uncorrelated noise; 3) stationary, correlated noise (whose covariance preferably becomes zero when the separation time increases beyond a certain limit); and 4) nonstationary, correlated noise (whose covariance preferably becomes zero when the separation time increases beyond a certain limit) The corresponding noise covariance matrix is 1) proportional to the identity matrix; 2) a diagonal matrix; 3) determined once the covariances in the first row are specified (and preferably has zero elements beyond a certain distance from the diagonal); and 4) a symmetric matrix (which preferably has zero elements beyond a certain distance from the diagonal)

This problem is usually solved by least squares, which provides an optimum solution only for stationary, uncorrelated noise For nonstationary and/or correlated noise, an iterative weighted least squares procedure is now described

Iterative Weighted Least Squares Procedure

The foregoing has shown that the general error model for an inertial navigator is a linear combination of system error sources X_1, X_2, , X_p with unknown coefficients β_1, β_2, , β_p Let the corresponding measurement Y of the system error be corrupted by noise ϵ in such a way that

$$Y = \sum_{j=1}^{p} \beta_j X_j + \epsilon \tag{10}$$

Given the observations (X_{1i}, X_{2i}, , X_{pi}, Y_i) for $i = 1, 2,$, n, the problem is to obtain estimators of β_1, β_2, , β_p and of

$$\sum_{j=1}^{p} \beta_j X_j$$

This estimation must be accomplished when ϵ_1, ϵ_2, , ϵ_n have mean zero and an n-by-n covariance matrix Σ, of variances σ_{ii} and covariances σ_{ik}, whose form alone is known

Suppose that W is an n-by-n matrix of weights w_{ik} Then the weighted least squares estimators of β_1, β_2, , β_p determined by W are those that minimize

$$\sum_{i=1}^{n} \sum_{k=1}^{n} w_{ik} \left(Y_i - \sum_{j=1}^{p} \beta_j X_{ji} \right) \left(Y_k - \sum_{j=1}^{p} \beta_j X_{jk} \right) \tag{11}$$

When W is the identity matrix, these estimators are called the least squares estimators

The iterative weighted least squares procedure 1) obtains the least squares estimators (because only the form of the noise covariance matrix is known); 2) using the data and these least squares estimators, obtains a matrix of estimators for the noise variances and covariances; 3) obtains the weighted least squares estimators that are determined by the inverse of this matrix; and 4) iteratively repeats steps 2 and 3 with the least squares estimators replaced by the latest set of weighted least squares estimators (proceeding

§ See Anderson and Bancroft[4] for a discussion of the statistical concepts and terms used in this paper

¶ In this paper, "stationary" means that the noise variance is constant and the noise covariance depends only upon the separation time, whereas "uncorrelated" implies that the noise covariances are zero toward the anticipated improvement of the matrix of estimators for the noise variances and covariances and, thereby, the weighted least squares estimators determined by the inverse of this matrix) until a preassigned level of improvement is attained

Stated symbolically, the foregoing procedure 1) obtains the least squares estimators $\hat{\beta}_1^{(0)}, \hat{\beta}_2^{(0)}, \quad, \hat{\beta}_p^{(0)}$; 2) using Y_i and

$$\hat{Y}_i^{(0)} = \sum_{j=1}^{p} \hat{\beta}_j^{(0)} X_{ji}$$

for $i = 1, 2, \quad, n$ and an appropriate estimation scheme, obtains an n-by-n matrix $\hat{\Sigma}^{(1)}$ of estimators $\hat{\sigma}_{ii}^{(1)}$ and $\sigma_{ik}^{(1)}$ for σ_{i_i} and σ_{ik}; 3) obtains the weighted least squares estimators $\hat{\beta}_1^{(1)}$, $\hat{\beta}_2^{(1)}, \quad, \hat{\beta}_p^{(1)}$ which are determined by $[\hat{\Sigma}^{(1)}]^{-1}$; and 4) iteratively repeats steps 2 and 3 with $\hat{\beta}_1^{(0)}, \hat{\beta}_2^{(0)}, \quad, \hat{\beta}_p^{(0)}$ replaced by $\hat{\beta}_1^{(\ )}, \hat{\beta}_2^{(\ )}, \quad, \hat{\beta}_p^{(\ )}$ and obtains $\hat{\Sigma}^{(\ +1)}$ and $\hat{\beta}_1^{(\ +1)}, \hat{\beta}_2^{(\ +1)}, \quad, \hat{\beta}_p^{(\ +1)}$ for $c = 1, 2, \quad, c^* - 1$, where c^* either is a preassigned constant or is determined when

$$\sum_{j=1}^{p} \left| \frac{\hat{\beta}_j^{(\ )} - \hat{\beta}_j^{(\ -1)}}{\hat{\beta}_j^{(\ )}} \right| \text{ or } \sum_{i=1}^{n} \left| \frac{\hat{Y}_i^{(\ )} - \hat{Y}_i^{(\ -1)}}{\hat{Y}_i^{(c)}} \right|$$

becomes less than some preassigned constant

An appropriate estimation scheme to use in step 2 is determined by the type of noise and the data When the observations are taken at different and equally spaced times, three such schemes are as follows:

1) With nonstationary, uncorrelated noise,

$$\hat{\sigma}_{ik}^{(1)} = \hat{\sigma}_{ki}^{(1)} = \begin{cases} \dfrac{1}{m+i} \displaystyle\sum_{q=-i+1}^{m} (Y_{i+q} - \hat{Y}_{i+q}^{(0)})^2 \text{ for } i = k = 1, 2, \quad, m \\ \dfrac{1}{2m+1} \displaystyle\sum_{q=-m}^{m} (Y_{i+q} - \hat{Y}_{i+q}^{(0)})^2 \\ \qquad \text{for } i = k = m+1, m+2, \quad, n-m \\ \dfrac{1}{n+m-i+1} \displaystyle\sum_{q=-m}^{n-i} (Y_{i+q} - \hat{Y}_{i+q}^{(0)})^2 \\ \qquad \text{for } i = k = n-m+1, n-m+2, \quad, n \\ 0 \text{ for } k = i+1, i+2, \quad, n, i = 1, 2, \quad, n \end{cases} \quad (12)$$

which is a suitably truncated (for $i = k = 1, 2, \quad, m$ and $n - m + 1, n - m + 2, \quad, n$) running average over $2m + 1$ points

2) With stationary, correlated noise for which the covariance becomes zero when the separation time increases beyond a certain limit (represented by K),

$$\hat{\sigma}_{ik}^{(1)} = \hat{\sigma}_{ki}^{(1)} = \begin{cases} \dfrac{1}{n-k+i} \displaystyle\sum_{q=1}^{n-k+i} (Y_q - \hat{Y}_q^{(0)})(Y_{q+k-i} - \hat{Y}_{q+k-i}^{(0)}) \\ \qquad \text{for } k = i, i+1, \quad, \min(i+K, n), i = 1, 2, \quad, n \\ 0 \text{ for } k = \min(i+K, n) + 1, \min(i+K, n) + \\ \qquad 2, \quad, n, i = 1, 2, \quad, n \end{cases} \quad (13)$$

3) With nonstationary, correlated noise, a proper combination of Eqs (12) and (13) can be used

In the foregoing, m should be selected small enough for the variance to be stable yet large enough for the estimator of the variance to be reliable, and K should be selected large enough for the covariance to become zero yet small enough for the estimator of the covariance to be reliable

Illustrative Example

To illustrate the effectiveness of the iterative weighted least squares procedure, an example was constructed from $n = 345$ values of $p = 5$ system error sources The corresponding 345 system errors,

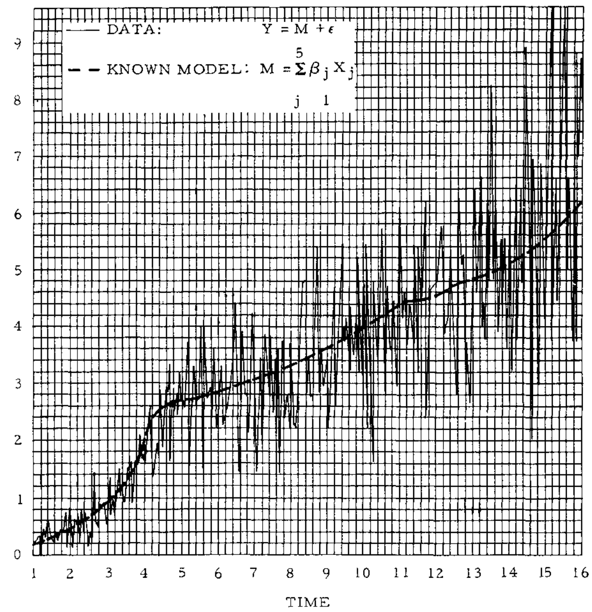

Fig 2 Data and known model

$$Y_i = 5X_{1i} - 50X_{2i} + 10X_{3i} + X_{4i} + 30X_{5i} + \epsilon_i$$

were generated by adding uncorrelated noise ϵ_i, with mean zero and variance $\sigma_{ii} = [0\ 1 + 1\ 6(i/345)]^2$, to the linear combination of X_{1i}, X_{2i}, X_{3i}, X_{4i}, and X_{5i} with known coefficients 5, −50, 10, 1, and 30, respectively

This illustrative example has the following properties: 1) the noise has variances of substantial size relative to the system eıror and substantial known variation with time and a known covariance matrix; 2) the system error sources are highly correlated among themselves (with correlation coefficients whose absolute values are above 0 87); and 3) the true values of β_1, β_2, β_3, β_4, and β_5 are known It is therefore not only a realistic and difficult example, but also one for which the answers and optimum estimators are known

Table 1 summarizes the information regarding β_1, β_2, β_3, β_4, and β_5 by listing 1) known values; 2) optimum estimators (i e , the weighted least squares estimators determined by the inverse of the known noise covariance matrix); 3) least squares estimators; and 4) five sets of iterative weighted least squares estimators obtained with $m = 5$

The graphs of Y and the known value $M = 5X_1 - 50X_2 + 10X_3 + X_4 + 30X_5$ of the model appear in Fig 2 The optimum estimators, least squares estimators, first set of iterative weighted least squares (IWLS) estimators, and fifth set of IWLS estimators may be compared by calculating the difference between the model estimator, based upon each such set of estimators, and the known model Figure 3 contains the graphs of these differences

Table 1 indicates that sufficient improvement in the estimators is attained by the third iteration In Fig 3, the IWLS model estimators (which are close to the optimum one) are superior to the least squares model estimator at both ends of the time scale and in the middle, except for two short intervals, of this scale The IWLS model estimators are, therefore, preferable to the least squares model estimator for extrapolation purposes

Discussion

A method has been presented which permits the dominant inertial system error sources to be estimated from flight test data The method applies an iterative weighted least squares procedure to the estimation of the unknown coefficients in the error model The method is not of the "cookbook" variety, but requires the exercise of engineering and statistical judgement for its use In particular, experience is required for determining the error sources to be admitted to the model Furthermore, after a solution is found, some check on its adequacy must be made

The authors have found no rule that will enable one to automatically determine the error sources to include in the model If there are only a small number of possible error sources, then one useful method consists of attempting to match the shape of the observed error propagation against the characteristic shapes for the possible error sources Otherwise, Efroymson's stepwise procedure,[6] which preselects the p most appropriate error sources from among the possible ones by means of a least squares technique, is quite useful

When a solution is obtained, the following checks on its adequacy can be made:

1) Goodness of fit of the estimators: the quality of

$$\hat{Y}^{(\ *)} = \sum_{=1}^{p} \hat{\beta}_j^{(c*)} X_j$$

as an estimator of Y and

$$\sum_{j=1}^{p} \beta_j X_j$$

2) Physical reasonableness of the estimators: the quality of $\hat{\beta}_1^{(c*)}, \hat{\beta}_2^{(\ *)}, \quad , \hat{\beta}_p^{(\ *)}$ as estimators of $\beta_1, \beta_2, \quad , \beta_p$ when compared with a priori information (Prior analysis of autonavigator performance has given an indication of the ensemble means, variances, and covariances of $\beta_1, \beta_2, \quad , \beta_p$ Very large differences between the estimators and the corresponding ensemble means, particularly if appearing in several sources simultaneously, should be viewed with suspicion)

3) Analysis of the residuals: the amount of trend exhibited by the graph of $Y - \hat{Y}^{(\ *)}$ vs time (Assuming a perfect representation of the error curve by the model, a plot of the residuals would exhibit a random pattern corresponding to the instrumentation noise In the event of inadequacies in the model, however, smooth trends in the residuals should result An analysis of the residuals in the light of the known propagation of the various error sources should give insight as to the source of inadequacy of the model)

The two 7090 or 7094 FORTRAN computer programs that have been written to implement this procedure for nonstationary, uncorrelated noise are NAWL1 and NAWL2 [7 8]** The heart of any least squares or weighted least squares procedure is the solution of the normal equations [i e , the p simultaneous linear equations in $\beta_1, \beta_2, \quad , \beta_p$ which result from the minimization of the expression in Eq (11)] by matrix inversion or other techniques When the system error sources $X_1, X_2, \quad , X_p$ are highly correlated, the merit of the solution to these normal equations is highly dependent upon the accuracy of the technique employed to calculate it Both NAWL1 and NAWL2 perform the calculations associated with the matrix inversion in double precision and, also, iterate on the calculated inverse matrix until it is sufficiently accurate NAWL1 is the basic program, whereas NAWL2 is modified to allow for Efroymson's stepwise procedure [6]

At each stage, the iterative weighted least squares procedure obtains the best possible estimators, based upon the available information at that stage Although this procedure is considerably more complicated than least squares, its use is quite

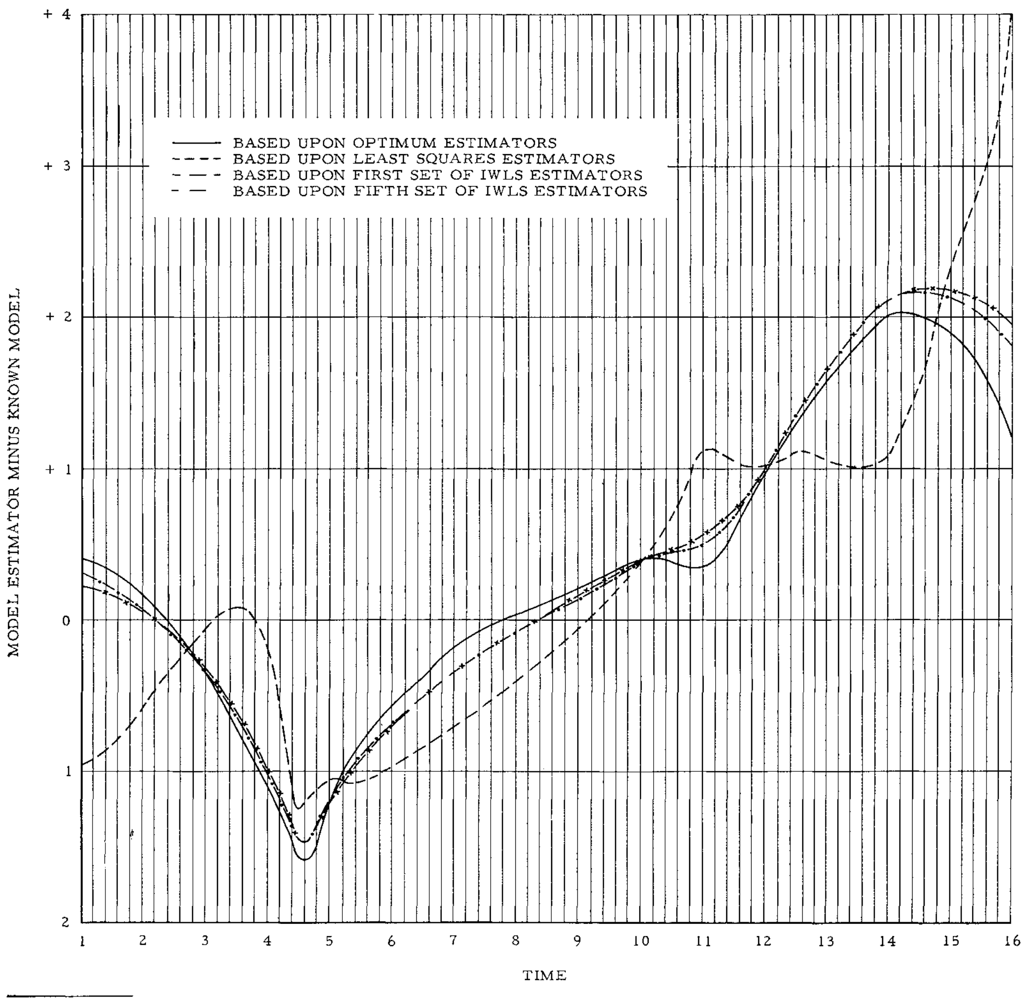

Fig 3 Comparison of estimators

** The authors wish to acknowledge the exceptionally fine computer programming done by P L Hsu and K V Smith

Table 1 Summary of information regarding β_1, β_2, β_3, β_4, and β_5

	β_1	β_2	β_3	β_4	β_5
Known values	5	−50	10	1	30
Optimum estimators	7 19580	74 2015	14 6472	2 08737	283 342
Least squares estimators	−10 5887	−248 760	14 1256	−34 9766	−172 545
First iterative weighted least squares estimators	6 74929	46 5394	12 8808	5 62012	229 349
Second iterative weighted least squares estimators	5 59523	29 9388	13 2030	1 35467	213 584
Third iterative weighted least squares estimators	5 60868	30 5758	13 2339	1 36987	214 643
Fourth iterative weighted least squares estimators	5 60542	30 5431	13 2359	1 35466	214 638
Fifth iterative weighted least squares estimators	5 60539	30 5450	13 2361	1 35484	214 641

practical if a large digital computer is available and is indicated whenever accuracy is at a premium

The procedure is intuitively quite appealing It is therefore not unreasonable to conjecture that it will converge (except, possibly, under pathological conditions) to a very good, if not near-optimum, solution to the problem However, its general (and in particular, its convergence) properties should be theoretically and empirically investigated Grenander and Rosenblatt,[9] Magness and McGuire,[10] and Golub[11] have studied the nature of, and difference between, the optimum and least squares (weighted least squares, for Golub) estimators Their results should be incorporated into this investigation

Sometimes, the same coefficient will appear in the inertial system error model for more than one coordinate axis Consequently, the iterative weighted least squares procedure has been modified to estimate the unknown coefficients in several related models simultaneously, by properly "stacking" the data

The second check on the results of the iterative weighted least squares procedure (i e , the physical reasonableness of the estimators) focuses attention on a somewhat alarming, and not-too-well-publicized, characteristic of both the least squares and weighted least squares techniques These techniques produce estimators $\hat{\beta}_1$, $\hat{\beta}_2$, , $\hat{\beta}_p$ which are not necessarily close to β_1, β_2, , β_p, respectively, in the same sense that

$$\sum_{j=1}^{p} \hat{\beta}_j X_j$$

is necessarily as close as possible to Y and, hopefully, to

$$\sum_{j=1}^{p} \beta_j X_j$$

Techniques have been derived which incorporate a priori information concerning the ensemble means, variances, and covariances of β_1, β_2, , β_p into the procedure by performing an additional iteration that uses this information

An alternative to Efroymson's stepwise procedure[6] has also been developed The alternative 1) partitions the set of possible error sources into subsets of "highly related" ones by using an appropriate technique; 2) uses an appropriate weighted average of the error sources in a subset as the representative error source for that subset; 3) obtains estimators of the unknown coefficients in the error model that is composed of only the representative error sources by using the iterative weighted least squares procedure; and 4) apportions the estimator of the unknown coefficient for the representative error source of a subset among the unknown coefficients for the error sources in that subset by using the same weighting scheme as the corresponding appropriate weighted average These extensions of the iterative weighted least squares procedure are currently being evaluated and will be described in a subsequent paper

Iterative procedures have also been suggested by Turner, Monroe, and Lucas[12] and by Fisher[13] for related, but less general, problems with nonstationary, uncorrelated noise Goodman[14] has presented a noniterative procedure, which solves the problem in the frequency domain rather than in the time domain for the case of stationary, correlated noise

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